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# Analyzing how inflation affects non-instantly decaying goods with demand linked to ads and selling price in a dual-warehouse setup

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ABSTRACT

This study explores inventory management strategies specifically designed for non-instantaneous deteriorating goods under inflationary conditions, utilizing a dual-warehouse system—one owned and the other rented with limited storage capacity. We examine the effects of advertising frequency and product selling price on demand rates, taking into account the gradual decline in customer patience, which leads to partial backlogging of shortages. The primary objective is to determine optimal replenishment policies for retailers that effectively minimize total costs per unit time. In real life, managing oil inventory is crucial for industries where factors like gradual deterioration, demand fluctuations due to advertising and pricing, and sensitivity to inflation require sophisticated inventory models to optimize replenishment policies and minimize costs. To validate our proposed inventory model, we provide a numerical. A sensitivity study using MATLAB R2024a software highlights the impact of parameter changes, providing significant information for decision-makers across various industries.

## 1. Introduction

In conventional inventory models, the focus is often on a single storage facility. However, in certain situations where there's a need to store a significant amount of stock, the existing single warehouse models prove inadequate. To address this, surplus units are stored in one or more additional warehouses (RW) through rental arrangements. Hartley [1] was among the pioneers to introduce the concept of this inventory model, featuring two warehouses (2WH). Sarma [2] later enhanced Hartley's model by integrating fixed transportation costs. Similarly, Jaggi et al. [3] presented a two-warehouse inventory model with partial backlog.

The underlying premise of these two-warehouse inventory models typically assumes infinite capacity for the rental warehouse (RW). However, this assumption is unrealistic, necessitating the use of additional warehouses. Since rental warehouses generally have higher holding costs compared to owned warehouses (OW), a more economical strategy is to initially fulfill orders from rental warehouses and subsequently distribute items from owned warehouses. Rangarajan and Karthikeyan [4] found that the deterioration and demand rates are piecewise continuous cubic functions of time. Several researchers, including Jaggi et al. [5], Rangarajan and Karthikeyan [6], Chakrabarty et al. [7], San-José et al. [8], and Cárdenas-Barrón et al. [9], have investigated numerous scenarios and contexts in the field of dual warehousing. They have explored different aspects and implications of this model, contributing to the broader understanding of its applications and effectiveness.

Since the pioneering work of Ghare and Schrader [10] in 1963, there has been a notable shift towards more intricate models in the study of deteriorating inventories. Covert and Philip [11] introduced a two-parameter distribution based on the Weibull distribution for deterioration, which was later expanded to a three-parameter distribution by

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Philip [12]. Non-instantaneous deterioration refers to the gradual decline in the quality or usability of items over time, rather than immediate spoilage. Tiwari et al. [13] developed particle swarm optimization for a multi-warehouse inventory model, taking into account non-instantaneous deteriorating items and inflation, acknowledging the gradual loss of freshness in many products over time. Rangarajan and Karthikeyan [14] contributed inventory models with linear deterioration rates and cubic demand rates for both non-instantaneous and instantaneous deteriorating products. Pervin et al. [15] proposed a sustainable inventories model for non-instantaneous decaying products, taking into account composite demand & environmental effect elements. Sundararajan et al. [16] looked at an EOQ model for non-instantaneous deteriorating items having price & time-dependent demand, along with the backlog.

In inventory modelling, the demand rate is typically assumed to be constant, time-dependent, and impacted by stock levels. Furthermore, inventory systems rely heavily on selling prices. Recent observations show that a product's selling price and social media marketing have a major impact on customer or buyer attractiveness. Rangarajan and Karthikeyan [17] proposed an EOQ inventory model that includes cubic demand, degradation rate, & time-dependent holding costs. Hesham and Chaithan [18] introduced a method based on price-dependent inventory, while Rastogi et al. [19] proposed two warehouse inventory techniques that take into account price-influenced demand as well as deterioration caused by partial backlog. Khan et al. [20] proposed a model for non-instantaneously deteriorating inventory that includes nonlinear stock-dependent demand, a hybrid payment approach, and partial backlog shortages. San-José et al. [21] examined an inventory system in which demand was determined by both time and price, taking into account backlogged shortages. Sundararajan et al. [22] developed an inventory system specifically designed for handling non-instantaneous deteriorating commodities, taking into consideration factors such as backlog and time discount.

Furthermore, the financial dynamics of numerous emerging economies, including but not confined to India, Bangladesh, and Brazil, undergo frequent and substantial shifts due to elevated inflation rates. The evolving financial circumstances emphasize the necessity of considering the impact of inflation, as it has become an integral aspect of the global economy. Buzacott [23] pioneered the presentation of an EOQ model with constant demand under the influence of inflation. Guria et al. [24] delved into an item's inventory strategy, considering immediate partial payments and adjustments in demand, purchasing, and selling due to inflation. Bhunia and Shaikh [25] explored two-warehouse (2WH) inventory issues within the context of an inflationary interval.

The fundamental goal of the proposed model is to enhance the overall cost-effectiveness of the system, thereby increasing revenue generation and ensuring customer satisfaction. This approach is particularly relevant in industries such as dairy, bakery, floristry, and oil, where maintaining multiple warehouses is crucial. Both theoretically and practically, there is a compelling need to investigate the impact of inflation on non-instantaneously deteriorating goods, considering demand influenced by advertisements and selling prices within a two-warehouse system, while accommodating partially backlogged shortages. Inflation introduces a dynamic element that can significantly alter inventory costs and valuations over time, thus affecting supply chain management comprehensively. Incorporating non-instantaneous deterioration addresses the realistic scenario of gradual item degradation, while integrating advertisement and selling price-dependent demand adds complexity by reflecting consumer behavior.

Udayakumar et al. [26] investigated an economic ordering method for non-instantaneous degrading commodities, where demand is influenced by prices and advertisements, and permissible payment delays are taken into account during inflationary situations. Similarly, Shah and Shah [27] created inventory strategies tailored to non-instantaneously deteriorating commodities, taking into consideration advance sales and advertising efforts. Furthermore, recent research reviews by Limi et al. [28] provide insights into the evolution of inventory models of non-instantaneous deteriorating objects, detailing advances made between 2006 and 2022.

Inventory management and fluid dynamics both optimize flow processes through principles of conservation and control, employing differential equations and simulation models for efficient system design. However, few studies have explored this intersection [29-37]. Managing oil inventory is critical for industries such as energy, manufacturing, and transportation, where the gradual deterioration of oil, demand fluctuations influenced by advertising and market prices, and sensitivity to inflation necessitate sophisticated inventory models to optimize replenishment policies and minimize operational costs as a real life application which has been discussed in Section 5.

The integration of these findings into inventory practices equips retailers with the tools to minimize costs, optimize inventory levels, and ensure long-term viability in the competitive retail landscape. Additionally, studies by Chandramohan et al. [38], De et al. [39], Pal et al. [40], Limi et al. [41], and Meena et al. [42] further contribute to the understanding of inventory management strategies, spanning from comprehensive inventory systems to addressing challenges posed by non-instantaneous deteriorating goods and fluctuating economic conditions. Additionally, the consideration of a two-warehouse system acknowledges the practical complexities of modern distribution networks.

Inflation exerts significant influence on inventory management strategies, impacting pricing decisions, production planning, and overall supply chain dynamics (Padiyar et al. [43]; De et al. [44]). Studies have explored optimal pricing and advertisement strategies for deteriorating items under inflationary pressures (Trivedi et al., [45]), imperfect production models considering inflation's impact on the time value of money (Padiyar et al. [46]), and integrated inventory models that account for inflationary effects on demand. These investigations underscore the necessity of adaptive inventory control mechanisms to mitigate inflation-related risks and enhance supply chain resilience in fluctuating economic environments. By investigating the implications of inflation on these multifaceted dynamics, the research aims to provide valuable insights for inventory management strategies, pricing policies, and advertisement decisions, contributing to the optimization of supply chain performance and operational efficiency in a real-world context. Fig. 1 exposes graphically the entire system. This flowchart illustrates the management of non-instantaneously deteriorating items with demand influenced by advertising and selling price, considering partially backlogged shortages. Purchased items are stored in rented and owned warehouses, with the effects of inflation impacting the process before reaching the retailer and consumer. First the products from the rented warehouse are been distributed then the items in the owned warehouse.

### 1.1. Research question

Based on the above discussion, our objective is to explore the following research questions:

i. How can optimal replenishment policies for non-instantaneously deteriorating goods be developed in a dual-warehouse system under inflationary conditions?
ii. What impact do advertisement frequency and product selling price have on the demand rates of these goods?
iii. How can total costs per unit time be minimized considering inflation, advertisement, and pricing factors?

### 1.2. Research objective and novelty of proposed work

The objective of this research is to develop and validate optimal replenishment policies for non-instantaneously deteriorating goods in a dual-warehouse system under inflationary conditions. The study aims to investigate the effects of advertisement frequency and product selling price on demand rates, taking into account the gradual decline in

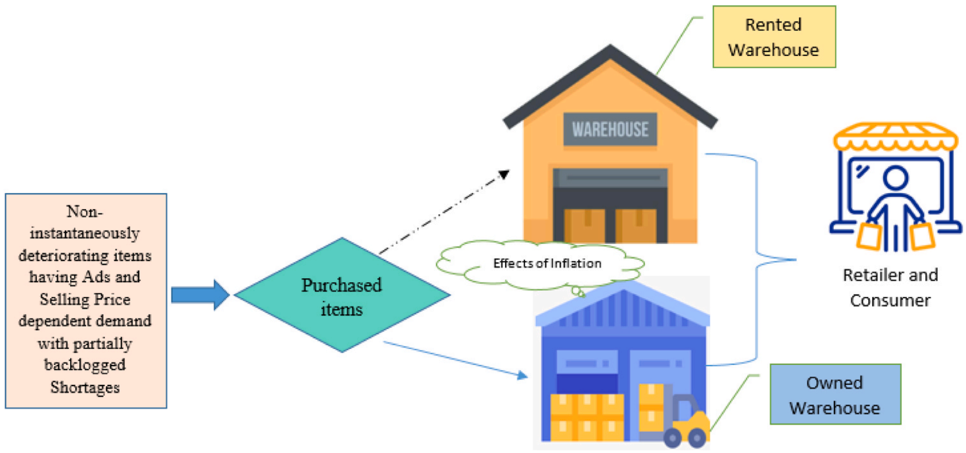


**Fig. 1.** Graphical representation of the proposed model.

customer patience and the resulting partial backlogging of shortages. By analyzing the interplay between inflation, advertising, pricing, and warehouse constraints, the research seeks to provide strategies for minimizing total costs per unit time, thereby offering valuable insights for decision-makers in various industries.

The novelty of this work lies in its comprehensive approach to managing non-instantaneously deteriorating inventory under inflationary conditions in a dual-warehouse system. By integrating advertising frequency and product selling price into the demand model, and using differential equations for optimization, the study provides a unique framework for minimizing total costs.

### *1.3. Orientation*

The subsequent sections of this paper are organized as follows: Section 2 provides the research gap, motivation and contribution of the proposed model. Section 3 elucidates the notations and fundamental assumptions employed in this study. Section 4 delves into the formulation and resolution of the mathematical model. Section 5 showcases the application of numerical examples to solve the model. Section 6 conducts sensitivity analysis using MATLAB-R2024a to assess the model's performance. Managerial perspectives are addressed in Section 7. Finally, Section 8 provides a summary of the article and suggests the direction of future research.

## 2. Research gap, motivation and contribution

### *2.1. Research gap*

The existing literature emerges from a lack of studies that specifically address the combined influence of advertisement and selling price dependence on inventory management for non-instantaneous deteriorating items within a dual-warehouse system under inflationary conditions, where fully backlogged shortages are allowed. While prior research has examined similar factors individually or within single-warehouse settings, there is a notable absence of studies that integrate these variables into a comprehensive analysis within the specified context. Thus, the present study aims to fill this gap by offering insights into optimal inventory management strategies tailored to these complex dynamics, providing valuable guidance for retailers navigating dynamic market environments.Table 1 outlines several research papers compared with the proposed model, highlighting the distinctive aspects that differentiate the study from existing literature.A comparative analysis of various studies on inventory management, highlighting differences in demand types, deterioration considerations, number of warehouses, shortage handling, and inflation effects.

### *2.2. Motivation*

This paper is motivated by the complex interactions among various factors that affect inventory management, especially concerning non-instantaneous deteriorating items in environments affected by inflation. Acknowledging the importance of advertising frequency and selling price on demand, along with the challenge of managing shortages in a dual-warehouse setup, the study aims to tackle the intricate challenges encountered by retailers. By comprehensively examining these dynamics, including the impact of inflation and fully backlogged shortages, the research aims to offer practical insights into optimal replenishment strategies that minimize costs and enhance operational efficiency. This endeavour is driven by the imperative for retailers to

**Table 1**
Illustrates a comparative analysis between the present study and earlier research efforts.

| Author | Demand | Deterioration | No. of Warehouses | Shortage | Inflation |
|---|---|---|---|---|---|
| Bhunia and Shaikh [25] | Interval environment | No | Two | Partial | Yes |
| Hesham and Chaithan [18] | Price-dependent | No | Single | No | No |
| Rastogi et al. [19] | Price-dependent | Deterioration | Two | Partial | No |
| Chakrabarty et al. [7] | Capacity constraints | Deterioration | Two | Partial | Yes |
| Rangarajan and Karthikeyan [6] | Ramp-type | Non-instantaneous | Two | Allowed | Yes |
| San-José et al. [21] | Time-and-price-dependent | No | Single | Allowed | No |
| Cárdenas-Barrón et al. [9] | Nonlinear stock dependent | No | Single | Partial | No |
| Sundararajan et al. [22] | Multivariate | Non-instantaneous | Single | Partial | Yes |
| Sundararajan et al. [16] | Price, time-dependent | Non-instantaneous | Single | Partial | Yes |
| Udayakumar et al. [26] | price and advertisement dependent | Non-instantaneous | Single | Partial | Yes |
| Khan et al. [20] | Nonlinear stock-dependent | Non-instantaneous | Single | Partial | No |
| Pervin et al. [15] | Composite | Non-instantaneous | Single | No | No |
| De et al.[44] | Price dependent | Deteriorating | One | Partial | Yes |
| Present paper | Advertisement and selling price dependent | Non-instantaneous | Two | Allowed | Yes |

adapt their inventory management practices in response to evolving market conditions, ultimately striving to improve profitability and competitiveness in a dynamic business landscape.

### 2.3. Contribution

The research significantly advances the field of inventory management by elucidating the nuanced interplay between advertisement and selling price dependence within a dual-warehouse system for non-instantaneous deteriorating items amidst inflationary conditions with fully backlogged shortages. By addressing this complex intersection, the study not only fills a critical gap in the existing literature but also provides actionable insights for both academic scholars and industry practitioners. Through the development and validation of a comprehensive inventory model, the research offers practical strategies for retailers to optimize their inventory management practices, thereby enhancing operational efficiency and reducing costs. Moreover, the findings contribute to a deeper understanding of how various factors interact to shape inventory decisions, facilitating the refinement of supply chain strategies and fostering innovation in the field. Overall, the research makes a substantial contribution by advancing knowledge, offering practical solutions, and guiding decision-making in inventory management within dynamic market environments.

## 3. Assumptions and notations

### 3.1. Notations

The following notations are utilized in constructing the model. Table 2 defines the notations for inventory parameters and variables, detailing their units and descriptions.

### 3.2. Assumptions

The mathematical model proposed in this paper is based on following assumptions:

1. There is an endless replenishment rate.
2. The lead time is 0.
3. The inventory model is developed for single item.
4. The negative inventory backlogging rate is represented as $e^{-\delta(T-t)}$, $0 \leq \delta \leq 1$, where δ is the backlogging parameter during the shortage period.
5. Both rented and owned warehouses have a limited amount of space.
6. First, in accordance with demand, RW's inventory will be depleted, and then OW will follow suit in accordance with LIFO dispatching policy.
7. Items begin to deteriorate after their lifespan. That is, there is no deterioration of the product during its specified period of time. It will thereafter continue to decline at the same rate.
8. The deteriorating items are not replaced or repaired during the cycle.
9. The holding cost fluctuates and is a linear function of time.
10. The frequency of advertisements(A) and the product's selling price (f(p)) both affect the demand rate.
    i.e., $D(A, p) = A^{\lambda} f(p)$
    Where $f(p) = (k - p) > 0$ and `k`is a constant.
11. Inflation is considered in the inventory systems with the rate r.

## 4. Mathematical model: methodology, formulation and solution

The study's methodology involves several key steps. First, we frame a mathematical model that incorporates non-instantaneous deterioration, inflation effects, dynamic demand influenced by advertising and selling price, and partial backlogging. We then derive the total cost formula, which includes holding costs, deterioration costs, inflation impacts, and backlogging costs. Using the algorithm detailed in Section 4.1, we solve the model for a numerical example by initializing parameters and decision variables, iteratively adjusting them to minimize the total cost function, and verifying the necessary criteria for optimality. Finally, we perform a sensitivity analysis using the obtained parameter values to examine how changes in key inventory system parameters affect the total cost and other performance measures, providing valuable insights for decision-makers.

Here an inventory model for non-instantaneous deteriorating items across two warehouses, adhering to the stated assumptions. The model permits shortages and incorporates lost sales during such instances. Fig. 2, the inventory time diagram, illustrates the dynamics of the inventory model at time 't'.This graph represents inventory levels over time, showing the combined effects of demand and deterioration for rented (RW) and owned (OW) warehouses. The inventory decreases due to demand (red dashed line) and deterioration (blue dashed line), with the total impact depicted by the purple line.

The inventory levels of the items in both Owned Warehouses (OW) and Rental Warehouses (RW) at time t during the period (0, T) can be

**Table 2**
Notations of inventory parameters and variables.

| *Notations* | *Units* | *Description* |
|---|---|---|
| $\alpha$ $\beta$ $r$ T $t_d$ L M N C H(t) G(t) $I_o(t)$ $I_r(t)$ $S$ | Constant | In OW, the deterioration rate, where $0 \leq \alpha \leq 1$. |
| L(t) OC TC $C_1$ | Constant | In RW, the deterioration rate, where $0 \leq \beta \leq 1$. |
| | Year | The rate of inflation. |
| | Year | The duration of the inventory cycle. |
| | Unit | The interval when no deterioration takes place. |
| | Unit | The highest level of inventory (the RW capacity). |
| | Unit | The amount ordered for each cycle of inventory. |
| | Unit | The quantity of OW. |
| | $/unit | The purchasing cost per unit time |
| | $/unit | The holding cost per unit time in OW, H(t)= at, a>0 |
| | $/unit | The holding cost per unit time in RW, G(t)= bt, b>0 |
| | $/unit | The inventory level in OW at time t. |
| | $/unit | The inventory level in RW at time t. |
| | $/unit | The cost incurred due to shortages per unit time. |
| | $/unit | The quantity of lost sales at any given time t, where $t_o \leq t \leq T$ The cost associated with placing orders |
| | $/unit | per unit time. |
| | $/unit | The overall system cost. |
| | | The cost per unit time attributed to lost sales when a shortage occurs. |
| | | *Decision Variable* |
| $t_r$ $t_o$ | Year | The duration when the inventory level in RW reaches zero. |
| | Year | The interval when the inventory level in OW reaches zero. |

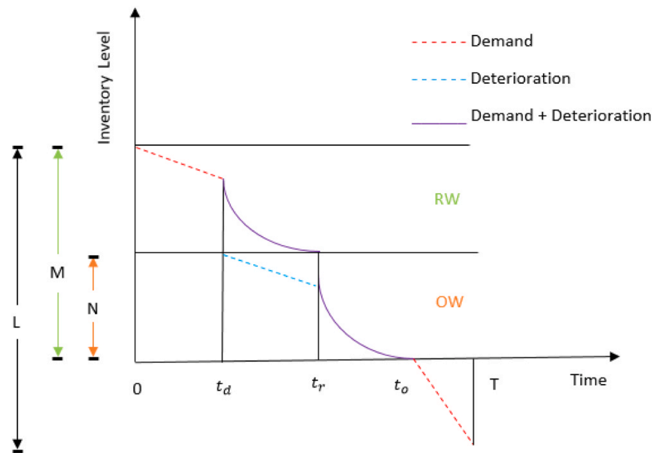


**Fig. 2.** Graphical representation of the two warehouse inventory system.

represented using the differential equations.

$$\frac{dI^{1R}(t)}{dt} = -A^{\lambda}f(p), \quad 0 \le t \le t_d \tag{1}$$

$$\frac{dI^{2R}(t)}{dt} + \beta I^{2R}(t) = -A^{\lambda}f(p), \ t_d \le t \le t_r \tag{2}$$

$$\frac{dI^{1O}(t)}{dt} = 0, \quad 0 \le t \le t_d \tag{3}$$

$$\frac{dI^{2O}(t)}{dt} + \alpha I^{2O}(t) = 0, \quad t_d \le t \le t_r \tag{4}$$

$$\frac{dI^{3O}(t)}{dt} + \beta I^{3O}(t) = -A^{\lambda}f(p), \ t_r \le t \le t_o \tag{5}$$

$$\frac{dI^{S}(t)}{dt} = -A^{\lambda}f(p)e^{-\delta(T-t)}, t_o \le t \le T \tag{6}$$

By using the specified boundary conditions to solve the aforementioned differential Eqs. (1) to (6), the inventory levels at different time periods are found.

$I^{1R}(0) = M - N$, $I^{2R}(t_r) = 0$, $I^{1O}(0) = N$, $I^{2O}(t_d) = N$, $I^{3O}(t_o) = 0$, $I^{S}(t_o) = 0$

Also at $t = t_d$, $I^{1R}(t) = I^{2R}(t)$

Eq. (1) follows with the condition at $t = 0$, $I^{1R}(t) = M - N$, we get

$$I^{1R}(t) = -A^{\lambda}f(p)t + M - N \tag{7}$$

Eq. (2) follows by the condition att $= t_r$, $I^{2R}(t) = 0$, we get

$$I^{2R}(t) = \frac{A^{\lambda}f(p)}{\beta}\left[e^{\beta(t_r - t)} - 1\right] \tag{8}$$

Eq. (3) follows by the condition at $t = 0$, $I^{1O}(t) = N$, we get

$$I^{1O}(t) = N \tag{9}$$

Eq. (4) follows by the condition at $t = t_d$, $I^{2O}(t) = N$, we get

$$I^{2O}(t) = Ne^{\alpha(t_d - t)} \tag{10}$$

Att $= t_O$, $I^{3O}(t) = 0$, we obtain by solving Eq. (5),

$$I^{3O}(t) = \frac{A^{\lambda}f(p)}{\alpha}\left[e^{\alpha(t_o - t)} - 1\right] \tag{11}$$

At $t = t_O$, $I^{S}(t) = 0$, we obtain by solving Eq. (6),

$$I^{S}(t) = \frac{-A^{\lambda}f(p)}{\delta}\left[e^{-\delta(T - t_o)} - e^{-\delta(T-t)}\right] \tag{12}$$

At $t = t_d$, $I^{1R}(t) = I^{2R}(t)$ and using (7) and (8) we have:

$$M = N + \frac{A^{\lambda}f(p)}{\beta}\left[\beta t_d + e^{\beta(t_r - t_d)} - 1\right] \tag{13}$$

Inventory shortage level during the interval $[t_o, T]$ is given by

$$IS = \frac{SA^{\lambda}f(p)}{\delta}\left[\frac{e^{-(T-t_o)}}{r}\left(e^{-rT} - e^{-rt_o}\right) - \frac{e^{-\delta T}}{(\delta + r)}\left(e^{(\delta+r)T} - e^{(\delta+r)t_o}\right)\right] \tag{14}$$

Therefore, during the replenishment cycle, the order quantity is ascertained using Eqs. (13) and (14) in the following manner:

$L = M + IS$

$$= N + \frac{A^{\lambda}f(p)}{\beta}\left[\beta t_d + e^{\beta(t_r - t_d)} - 1\right] + \frac{SA^{\lambda}f(p)}{\delta}\left[\begin{array}{c} \frac{e^{-(T-t_o)}}{r}\left(e^{-rT} - e^{-rt_o}\right) \\ -\frac{e^{-\delta T}}{(\delta + r)}\left(e^{(\delta+r)T} - e^{(\delta+r)t_o}\right) \end{array}\right] \tag{15}$$

The following elements make to the inventory system's overall cost per unit of time:

1. Ordering cost per cycle $= Ae^{-rT}$ (16)
2. Cost of inventory holding per cycle in RW

$$= \int_0^{t_d} atI^{1R}(t)e^{-rt}dt + \int_{t_d}^{t_r} atI^{2R}(t)e^{-rt}dt$$

$$= a\left[\left(A^{\lambda}f(p)\right)\left[e^{-rt_d}\left(\frac{t_d^2}{r} + \frac{2t_d}{r^2} + \frac{2}{r^3}\right) - \frac{2}{r^3}\right] + (M - N)\left[\frac{1}{r^2} - e^{-rt_d}\left(\frac{t_d}{r} + \frac{1}{r^2}\right)\right] + \frac{A^{\lambda}f(p)}{\beta}\left[e^{-rt_r}\left(\frac{-t_r}{r+\beta} - \frac{1}{(r+\beta)^2} - \frac{t_r}{r} + \frac{1}{r^2}\right) + e^{\beta t_r - t_d(r+\beta)}\left(\frac{t_d}{r+\beta} + \frac{1}{(r+\beta)^2}\right) + e^{-rt_d}\left(\left(\frac{t_d}{r} + \frac{1}{r^2}\right)\right)\right]\right] \tag{17}$$

3. The cost of inventory holding per cycle in OW

$$= \int_0^{t_d} btI^{1O}(t)e^{-rt}dt + \int_{t_d}^{t_r} btI^{2O}(t)e^{-rt}dt + \int_{t_r}^{t_o} btI^{3O}(t)e^{-rt}dt$$

$$= b\left[\frac{N}{r^2}\left(1 - t_d re^{-rt_d} - e^{-rt_d}\right) + N\left[\begin{array}{c} e^{-rt_d}\left(\frac{t_d}{(\alpha+r)} + \frac{1}{(\alpha+r)^2}\right) \\ -e^{-\alpha t_d - t_r(\alpha+r)}\left(\frac{t_r}{(\alpha+r)} + \frac{1}{(\alpha+r)^2}\right) \end{array}\right] + \frac{A^{\lambda}f(p)}{\alpha}\left[\begin{array}{c} e^{\alpha t_o - t_r(\alpha+r)}\left(\frac{t_r}{(\alpha+r)} + \frac{1}{(\alpha+r)^2}\right) \\ -e^{-rt_o}\left(\frac{t_o}{(\alpha+r)} + \frac{1}{(\alpha+r)^2}\right) \end{array}\right]\right] \tag{18}$$

4. The deteriorating cost per cycle in RW

$$= C\beta\int_{t_d}^{t_r} I^{2R}(t)e^{-rt}dt$$

$$= CA^{\lambda}f(p)\left[\frac{1}{(\beta+r)}\left(e^{\beta t_r - t_d(\beta+r)} - e^{-rt_r}\right) + \frac{1}{r}\left(e^{-rt_r} - e^{-rt_d}\right)\right] \tag{19}$$

5. The deteriorating cost per cycle in OW

$$= C\alpha\int_{t_d}^{t_r} I^{2O}(t)e^{-rt}dt + C\alpha\int_{t_d}^{t_r} I^{3O}(t)e^{-rt}dt$$

$$= C\left[\begin{array}{c}\frac{N\alpha}{(\alpha+r)}\left(e^{-rt_d} - e^{-\alpha t_d - t_r(\alpha+r)}\right)\\ +A^{\lambda}f(p)\left[\begin{array}{c}\frac{1}{(\alpha+r)}\left(e^{\alpha t_0 - t_r(\alpha+r)} - e^{-rt_o}\right)\\ +\frac{1}{r}\left(e^{-rt_o} - e^{-rt_r}\right)\end{array}\right]\end{array}\right] \tag{20}$$

6. The cost per cycle of an inventory shortage

$$= S\int_{t_o}^{T} -I^{S}(t)e^{-rt}dt$$

$$= \frac{SA^{\lambda}f(p)}{\delta}\left[\begin{array}{c}\frac{e^{-(T-t_o)}}{r}\left(e^{-rT} - e^{-rt_o}\right)\\ -\frac{e^{-\delta T}}{(\delta+r)}\left(e^{(\delta+r)T} - e^{(\delta+r)t_o}\right)\end{array}\right] \tag{21}$$

7. The Opportunity Cost for Each Cycle of Lost Sales

$$= C_1\int_{t_o}^{T} A^{\lambda}f(p)\left(1 - e^{-\delta(T-t)}\right)e^{-rt}dt$$

$$= C_1A^{\lambda}f(p)\left[\begin{array}{c}\frac{1}{r}\left(e^{-rt_o} - e^{-rT}\right)\\ -\frac{1}{(\delta+r)}\left(e^{-rT} - e^{-\delta T - t_o(\delta+r)}\right)\end{array}\right] \tag{22}$$

The comprehensive relevant inventory cost per unit over time is indicated and furnished by:

$$TC^{*} = \left[\begin{array}{c}\text{Ordereing cost} + \text{Holding cost RW} + \text{Holding cost OW}\\ +\text{Deteriorating cost RW} + \text{Deteriorating cost OW}\\ +\text{Shortage cost} + \text{Opportunity cost}\end{array}\right] \tag{23}$$

Substituting Eqs. (16) to (22) in Eq. (23) we have

$$TC^{*} = \left[\begin{array}{c}
Ae^{-rT} + \\
a\left[\begin{array}{c}\left(A^{\lambda}f(p)\right)\left[e^{-rt_d}\left(\frac{t_d^2}{r} + \frac{2t_d}{r^2} + \frac{2}{r^3}\right) - \frac{2}{r^3}\right] + (M-N)\left[\frac{1}{r^2} - e^{-rt_d}\left(\frac{t_d}{r} + \frac{1}{r^2}\right)\right]\\ +\frac{A^{\lambda}f(p)}{\beta}\left[\begin{array}{c}e^{-rt_r}\left(\frac{-t_r}{r+\beta} - \frac{1}{(r+\beta)^2} - \frac{t_r}{r} + \frac{1}{r^2}\right) + e^{\beta t_r - t_d(r+\beta)}\\ \left(\frac{t_d}{r+\beta} + \frac{1}{(r+\beta)^2}\right) + e^{-rt_d}\left(\left(\frac{t_d}{r} + \frac{1}{r^2}\right)\right)\end{array}\right]\end{array}\right]\\
+b\left[\begin{array}{c}\frac{N}{r^2}\left(1 - t_d re^{-rt_d} - e^{-rt_d}\right) + N\left[\begin{array}{c}e^{-rt_d}\left(\frac{t_d}{(\alpha+r)} + \frac{1}{(\alpha+r)^2}\right)\\ -e^{-\alpha t_d - t_r(\alpha+r)}\left(\frac{t_r}{(\alpha+r)} + \frac{1}{(\alpha+r)^2}\right)\end{array}\right]\\ +\frac{A^{\lambda}f(p)}{\alpha}\left[e^{\alpha t_0 - t_r(\alpha+r)}\left(\frac{t_r}{(\alpha+r)} + \frac{1}{(\alpha+r)^2}\right) - e^{-rt_o}\left(\frac{t_o}{(\alpha+r)} + \frac{1}{(\alpha+r)^2}\right)\right]\end{array}\right]\\
+CA^{\lambda}f(p)\left[\frac{1}{(\beta+r)}\left(e^{\beta t_r - t_d(\beta+r)} - e^{-rt_r}\right) + \frac{1}{r}\left(e^{-rt_r} - e^{-rt_d}\right)\right]\\
+C\left[\frac{N\alpha}{(\alpha+r)}\left(e^{-rt_d} - e^{-\alpha t_d - t_r(\alpha+r)}\right) + A^{\lambda}f(p)\left[\begin{array}{c}\frac{1}{(\alpha+r)}\left(e^{\alpha t_0 - t_r(\alpha+r)} - e^{-rt_o}\right)\\ +\frac{1}{r}\left(e^{-rt_o} - e^{-rt_r}\right)\end{array}\right]\right]\\
+\frac{SA^{\lambda}f(p)}{\delta}\left[\frac{e^{-(T-t_o)}}{r}\left(e^{-rT} - e^{-rt_o}\right) - \frac{e^{-\delta T}}{(\delta+r)}\left(e^{(\delta+r)T} - e^{(\delta+r)t_o}\right)\right]\\
+C_1A^{\lambda}f(p)\left[\frac{1}{r}\left(e^{-rt_o} - e^{-rT}\right) - \frac{1}{(\delta+r)}\left(e^{-rT} - e^{-\delta T - t_o(\delta+r)}\right)\right]
\end{array}\right] \tag{24}$$

Therefore, the restricted optimization problem that corresponds to it can be expressed as follows:

**Problem.** Minimise TC $= \frac{TC^*}{T}$

Subject to $0 \leq t_d \leq t_r \leq t_o \leq T$ (25)

The problem mention above can be addressed by employing the subsequent algorithm.

### 4.1. Solution procedure

Algorithms to solve the aforementioned issue.

**Step 1** Input the values for the specified parameters in the suggested inventory model: b, N, r, $t_d$, α,λ, M, A, β, f(p), C, δ, S, T, $C_1$ and a.

**Step 2** From the equation (25), find

$$\frac{\partial TC}{\partial t_r} = 0 and \frac{\partial TC}{\partial t_o} = 0$$

**Step 3** Constrained optimization problem stated above should be solved, and store the optimal value of $t_o$, $t_r$.

**Step 4** To minimize TC, solve the Hessian Matrix H

$$H = \begin{bmatrix} \frac{\partial^2 TC}{\partial {t_r}^2} & \frac{\partial^2 TC}{\partial t_r \partial t_o} \\ \frac{\partial^2 TC}{\partial t_o \partial t_r} & \frac{\partial^2 TC}{\partial {t_o}^2} \end{bmatrix}, \text{ Where } \frac{\partial^2 TC}{\partial {t_r}^2} > 0 \ \frac{\partial^2 TC}{\partial {t_o}^2} > 0 \& \begin{bmatrix} \frac{\partial^2 TC}{\partial {t_r}^2} & \frac{\partial^2 TC}{\partial t_r \partial t_o} \\ \frac{\partial^2 TC}{\partial t_o \partial t_r} & \frac{\partial^2 TC}{\partial {t_o}^2} \end{bmatrix}$$

$> 0$ are the minors.

**Step 5** The values of $t_r and t_o$ are got, using these values we find the optimal total cost.

## 5. Numerical analysis

In a real-life application of the proposed inventory model, let's consider an oil distribution company facing the challenge of managing refined oil inventory across multiple warehouses. Implementing the model's optimal replenishment times obtained from MATLAB R2024a (online), the inventory manager aims to streamline inventory management while minimizing costs. By carefully considering factors such as deterioration rates, shortage costs, and warehouse capacities, the company achieves significant cost savings and improved supply chain efficiency.

**Table 3**
Optimal Solution for the models got using MATLAB R2024a.

| Model | Description | $t_r$ | $t_o$ | Optimal Total Cost |
|---|---|---|---|---|
| 1 | Non-instantaneous deterioration items with shortages, partially backlogged | 3.3612 | 5.8706 | $181,870 |
| 2 | Instantaneous deterioration items with shortages, partially backlogged | 3.2529 | 5.6304 | $190,880 |
| 3 | Non-instantaneous deterioration items without shortages, partially backlogged | 3.3708 | 5.6312 | $216,640 |
| 4 | Instantaneous deterioration items without shortages, partially backlogged | 3.5248 | 5.8719 | $226,880 |
| 5 | Non-instantaneous deterioration items with shortages, completely backlogged | 3.3612 | 5.8606 | $182,020 |
| 6 | Instantaneous deterioration items with shortages, completely backlogged | 3.1911 | 5.6300 | $191,170 |
| 7 | Non-instantaneous deterioration items without shortages, completely backlogged | 3.5433 | 5.8594 | $229,770 |
| 8 | Instantaneous deterioration items without shortages, completely backlogged | 3.4765 | 5.6198 | $236,460 |

In this section, we apply the proposed inventory model to a real-life scenario in the oil industry. Utilizing numerical values based on cost variations observed in real-time data from an oil distribution company.

Consider an inventory system with the following data in appropriate units to demonstrate the aforesaid solution procedure:

Parameters for each model:

For fixed values of b = \$0.5 per unit, a = 0.7, p = \$5 per unit, k = \$55 per unit, N = 10 units, r = 0.06, α = 0.6, λ = 1, A = \$3 per unit, β = 0.4, f (p) = \$50 per unit, M = 20 units, S = \$2 per unit, C_1 = \$0.5 per unit, C = \$2 per unit, T = 7, $t_d$ = 1.5.

Take δ = 0.9 for partially backlogged models 1–4 and δ = 1 for completely backlogged models 5–8. Instantaneous model we take $t_d$= 0 for Models 2, 4, 6, 8. The model without shortages can be got using $t_o$= T for Models 3, 4, 7, 8. Optimal solution for partially/completely backlogged models with or without shortages for instantaneous/non-instantaneous deteriorating items has been presented in Table 3.

These results provide insights into the optimal time for replenishment and order placement, as well as the associated total costs for each inventory model scenario.

### 5.1. Observations

The numerical examples provided the following results.

1. Among all the models considered, Model 1 stands out with the lowest total cost. Specifically, this inventory model designed for non-instantaneous deteriorating items, incorporating partially backlogged shortages, yields the most cost-effective solution compared to other models.
2. The total average cost achieved by Model 1 is notably 0.082 % lower than that of Model 5. This slight but significant difference underscores the superior cost efficiency of Model 1 over Model 5.
3. Similarly, Model 2 demonstrates superior cost performance compared to Model 6, with an optimum total average cost that is 0.152 % lower. This indicates that Model 2 offers a more economical solution in managing non-instantaneous deteriorating items with the considered parameters.
4. Model 3 exhibits remarkable cost savings, with its optimal total average cost being 4.051 % less than that of Model 7. This substantial difference highlights the efficiency gains achieved by Model 3 in mitigating costs associated with inventory management.
5. Model 4 outperforms Model 8 by having a 5.714 % lower optimal total average cost. This significant margin emphasizes the considerable cost advantages offered by Model 4 in comparison to Model 8, further supporting its efficacy in optimizing inventory-related expenses.

In essence, these results highlight the significance of choosing the

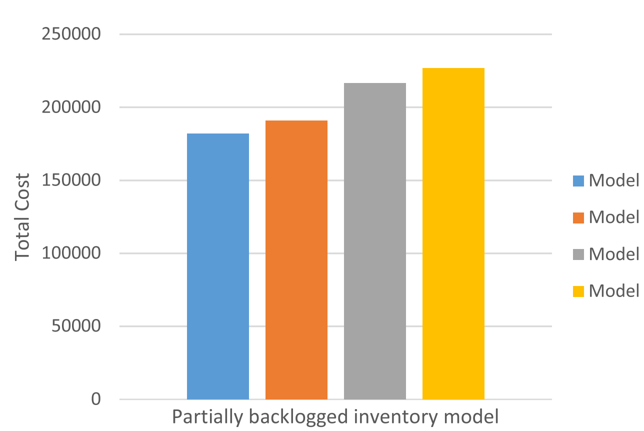


**Fig. 3.** Comparison of partially backlogged inventory model vs total cost.

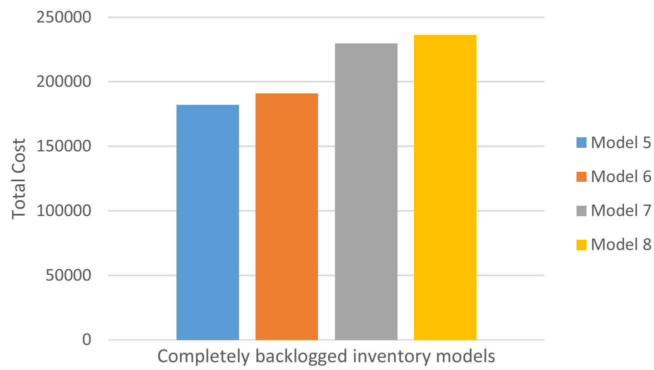


**Fig. 4.** Comparison of completely backlogged inventory model vs total cost.

right inventory model that aligns with particular operational needs and parameters to attain the most efficient cost-effectiveness in inventory management strategies. Upon examining it becomes apparent that among the initial four models, model 1 stands out as the most efficient, boasting the lowest total inventory cost. Fig. 3 compares the total inventory cost for the first four models that incorporate partially backlogged shortages. Similarly, Fig. 4 presents a comparison of the subsequent four models having completely backlogged shortages, revealing that model 5 emerges with the least total inventory cost.

## 6. Sensitivity analysis

The sensitivity analysis conducted on each parameter within the proposed model reveals that modifying these parameters significantly impacts the optimal values of the total system cost (TC). Notably, the total system cost experiences substantial fluctuations when the values of various parameters are adjusted by −50 %, −25 %, +25 %, and +50 %. Table 4 presents a summary of the sensitivity analysis, highlighting the impact of various parameters on the overall system performance. This analysis helps identify key factors influencing inventory cost.

### *6.1. Observations from sensitivity analysis*

The sensitivity analysis with the parameter values notes the following properties, which are based on Table 3.

1. As the parameter increases, the overall inventory cost decreases gradually. The total cost is moderately sensitive to the rate of deterioration in the owned warehouse.
2. When the parameter is increased, the overall inventory cost decreases. In other words, as the degradation of the rented warehouse increases, so does the entire cost. Notably, the rate of deterioration in the rented warehouse is quite sensitive, implying that even minor changes in this parameter have a large impact on the entire inventory cost.
3. An increase in the holding cost parameters "a" and "b" results in an increase in the overall cost. The parameter "b" exhibits moderate sensitivity, while the parameter "a" is highly sensitive. This suggests that changes in the parameter "a" have a more pronounced impact on the overall cost compared to changes in the parameter "b" in the context of holding costs.
4. As the parameters M and N gradually increase, the total inventory cost (TC) rises as the capacity of the owned warehouse (OW) increases. In contrast, the TC decreases when the capacities of both the owned warehouse (OW) and rented warehouse (RW) are increased simultaneously. Notably, both parameters M and N exhibit high sensitivity, implying that even small changes in these parameters significantly influence the total inventory cost, particularly in relation to warehouse capacities.
5. With a rise in the rate of inflation, parameter r the total cost decreases. The parameter "r" is extremely sensitive, implying that even minor changes in the rate of inflation can have a significant impact on the final cost. This shows a dynamic relationship in which inflation has a significant impact on the total cost in the overall context of the model.
6. The frequency of advertisement (parameter A) and $\lambda$ exhibit high sensitivity, meaning that as the values of these parameters increase, the total cost also experiences a significant and drastic increase. This underscores the considerable influence that the frequency of advertisement and the parameter $\lambda$ have on the overall cost in the context of the model.
7. When the non-instantaneous deterioration time $t_d$increases, total cost of the model decreases. This relationship is moderately sensitive, indicating that changes in the non-instantaneous deterioration time have a discernible but not extremely pronounced impact on the overall cost.
8. As the overall cycle time T is increased, the total cost decreases. The parameter T exhibits high sensitivity, suggesting that even small changes in the total cycle time can lead to a significant impact on the overall cost. This emphasizes the importance of considering and optimizing the total cycle time to achieve cost efficiencies in the model.
9. When the shortage cost (S) is increased, the total cost (TC) falls substantially. Simultaneously, the purchasing cost (C) increases, leading to a consequent decrease in the overall cost, and this relationship is moderately sensitive. These findings underscore the trade-offs between shortage costs and purchasing costs, indicating that adjustments in these parameters can impact the total cost in a moderately sensitive manner.
10. When the parameter $\delta$ is increased, the TC increases. The lost sale parameter $C_1$ value increases then the total cost increase and both parameters are moderately sensitive.
11. As the value of the parameter f(p) increases, the overall cost increases in a highly sensitive manner. This suggests that even small changes in the parameter f(p) can lead to a substantial impact on the overall cost in the model. The high sensitivity emphasizes the importance of carefully considering and managing this parameter to optimize the total cost in the given context.

Analyzing the total cost in relation to various parameters provides valuable insights into the factors influencing overall system costs. Fig. 5 illustrates the impact of different parameters on the total cost, highlighting the relationships and dependencies that affect inventory management efficiency.

## 7. Managerial insights

i. **Optimizing warehouse management**: Sensitivity analysis reveals that total inventory cost is influenced by factors such as deterioration rates, holding expenses, and warehouse capacities. Managers need to monitor and fine-tune these parameters meticulously to reduce costs while maintaining optimal warehouse functionality. Proper management of deterioration rates, efficient use of space, and balanced capacities between owned and rented warehouses can significantly enhance cost savings and efficiency.
ii. **Understanding inflation impact**: The sensitivity of total cost to inflation rates underscores the importance of anticipating and adapting to economic changes. Managers should develop strategies like adjusting pricing, renegotiating supplier contracts, and proactive financial planning to mitigate inflation impacts. These actions help maintain profitability and ensure business sustainability in an inflationary environment.
iii. **Advertisement strategy**: The frequency and effectiveness of advertisements significantly influence overall costs, highlighting

**Table 4**
Sensitivity analysis for the value considering different parameters.

| Parameter | Initial Value | % variation | $t_r$ | $t_o$ | Total Inventory Cost | Change in TC | Ratio | % change in significance |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | 0.6 | −50 | 1.6722 | 5.8758 | 190820 | 8950 | 0.047 | 4.69 |
| | | −25 | 2.5373 | 5.8725 | 186150 | 4280 | 0.023 | 2.30 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 4.0601 | 5.8709 | 179280 | −2590 | −0.014 | −1.44 |
| | | +50 | 4.5005 | 5.8701 | 177780 | −4090 | −0.023 | −2.30 |
| $\beta$ | 0.4 | −50 | 3.9223 | 6.4625 | 356380 | 174510 | 0.490 | 48.97 |
| | | −25 | 3.7651 | 6.0214 | 239720 | 57850 | 0.241 | 24.13 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 2.8797 | 5.1045 | 147490 | −34380 | −0.233 | −23.31 |
| | | +50 | 2.5161 | 4.5734 | 124040 | −57830 | −0.466 | −46.62 |
| N | 10 | −50 | 3.4444 | 5.8926 | 272340 | 90470 | 0.332 | 33.22 |
| | | −25 | 3.4381 | 5.8841 | 227100 | 45230 | 0.199 | 19.92 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.4134 | 5.8499 | 136630 | −45240 | −0.331 | −33.11 |
| | | +50 | 3.3823 | 5.8082 | 109510 | −72360 | −0.661 | −66.08 |
| r | 0.06 | −50 | 4.0586 | 6.1971 | 770290 | 588420 | 0.764 | 76.39 |
| | | −25 | 4.0802 | 6.3932 | 330160 | 148290 | 0.449 | 44.91 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 2.9562 | 5.4998 | 114340 | −67530 | −0.591 | −59.06 |
| | | +50 | 2.5912 | 5.2207 | 100113 | −81757 | −0.817 | −81.66 |
| $\lambda$ | 1 | −50 | 3.4286 | 5.8711 | 105000 | −76870 | −0.732 | −73.21 |
| | | −25 | 3.4286 | 5.8711 | 138190 | −43680 | −0.316 | −31.61 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.4286 | 5.8711 | 239350 | 57480 | 0.240 | 24.02 |
| | | +50 | 3.4286 | 5.8711 | 315000 | 133130 | 0.423 | 42.26 |
| b | 0.5 | −50 | 4.1157 | 5.8701 | 178920 | −2950 | −0.0165 | −1.65 |
| | | −25 | 3.6951 | 5.8704 | 180610 | −1260 | −0.0070 | −0.70 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.2437 | 5.8721 | 182850 | 980 | 0.0054 | 0.54 |
| | | +50 | 3.1075 | 5.8731 | 183660 | 1790 | 0.0097 | 0.97 |
| $\delta$ | | −50 | 4.1627 | 5.8782 | 178760 | −3110 | −0.0174 | −1.74 |
| | 0.9 | −25 | 3.7022 | 5.8734 | 180710 | −1160 | −0.0064 | −0.64 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.3064 | 5.8702 | 182410 | 540 | 0.0030 | 0.30 |
| | | +50 | 3.2557 | 5.8699 | 182640 | 770 | 0.0042 | 0.42 |
| f(p) | 50 | −50 | 3.4286 | 5.8711 | 129320 | −52550 | −0.406 | −40.64 |
| | | −25 | 3.4286 | 5.8711 | 136400 | −45470 | −0.333 | −33.34 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.4286 | 5.8711 | 227330 | 45460 | 0.200 | 20.00 |
| | | +50 | 3.4286 | 5.8711 | 272800 | 90930 | 0.333 | 33.33 |
| Parameter | Initial Value | % variation | $t_r$ | $t_o$ | Total Inventory Cost | Change in TC | Ratio | % change in significance |
| T | 7 | −50 | 3.3791 | 5.8708 | 365200 | 183330 | 0.5020 | 50.20 |
| | | −25 | 3.7403 | 5.8738 | 240710 | 58840 | 0.2444 | 24.44 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.2796 | 5.8701 | 145520 | −36350 | −0.2498 | −24.98 |
| | | +50 | 3.2354 | 5.8697 | 121590 | −60280 | −0.4958 | −49.58 |
| S | 2 | −50 | 3.3198 | 5.8703 | 182340 | 470 | 0.0026 | 0.26 |
| | | −25 | 3.3731 | 5.8707 | 182110 | 240 | 0.0013 | 0.13 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.4864 | 5.8716 | 181620 | −250 | −0.0014 | −0.14 |
| | | +50 | 3.5467 | 5.8721 | 181360 | −510 | −0.0028 | −0.28 |
| A | 3 | −50 | 3.4286 | 5.8711 | 129320 | −52550 | −0.4064 | −40.64 |
| | | −25 | 3.4286 | 5.8711 | 136400 | −45470 | −0.3334 | −33.34 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.4286 | 5.8711 | 227330 | 45460 | 0.2000 | 20.00 |
| | | +50 | 3.4286 | 5.8711 | 272800 | 90930 | 0.3333 | 33.33 |
| C | 2 | −50 | 2.8149 | 5.9049 | 184670 | 2800 | 0.0152 | 1.52 |
| | | −25 | 3.1661 | 5.8879 | 183020 | 1150 | 0.0063 | 0.63 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.6336 | 5.8545 | 181020 | −850 | −0.0047 | −0.47 |
| | | +50 | 3.7985 | 5.8381 | 180370 | −1500 | −0.0083 | −0.83 |
| $t_d$ | 1.5 | −50 | 3.2832 | 5.6721 | 186790 | 4920 | 0.026 | 2.63 |
| | | −25 | 3.3445 | 5.7564 | 184400 | 2530 | 0.014 | 1.37 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.5318 | 6.0105 | 179220 | −2650 | −0.015 | −1.48 |
| | | +50 | 3.6514 | 6.1702 | 176500 | −5370 | −0.030 | −3.04 |
| a | 0.7 | −50 | 3.3828 | 5.8023 | 117290 | −64580 | −0.551 | −55.06 |
| | | −25 | 3.4131 | 5.8499 | 136630 | −45240 | −0.331 | −33.11 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.4381 | 5.8841 | 227100 | 45230 | 0.199 | 19.92 |
| | | +50 | 3.4444 | 5.8926 | 272330 | 90460 | 0.332 | 33.22 |

(continued on next page)

**Table 4** (*continued*)

| Parameter | Initial Value | % variation | $t_r$ | $t_o$ | Total Inventory Cost | Change in TC | Ratio | % change in significance |
|---|---|---|---|---|---|---|---|---|
| $C_1$ | 0.5 | −50 | 3.5815 | 5.8724 | 181210 | −660 | −0.004 | −0.36 |
| | | −25 | 3.5069 | 5.8718 | 181530 | −340 | −0.002 | −0.19 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.3462 | 5.8705 | 182230 | 360 | 0.002 | 0.20 |
| | | +50 | 3.2592 | 5.8699 | 182630 | 760 | 0.004 | 0.42 |
| M | 20 | −50 | 3.3674 | 5.7963 | 92320 | −89550 | −0.970 | −97.00 |
| | | −25 | 3.3823 | 5.8082 | 106020 | −75850 | −0.715 | −71.54 |
| | | 100 | 3.4286 | 5.8711 | 181870 | - | - | - |
| | | +25 | 3.4444 | 5.8926 | 272340 | 90470 | 0.332 | 33.22 |
| | | +50 | 3.4524 | 5.9034 | 362810 | 180940 | 0.499 | 49.87 |

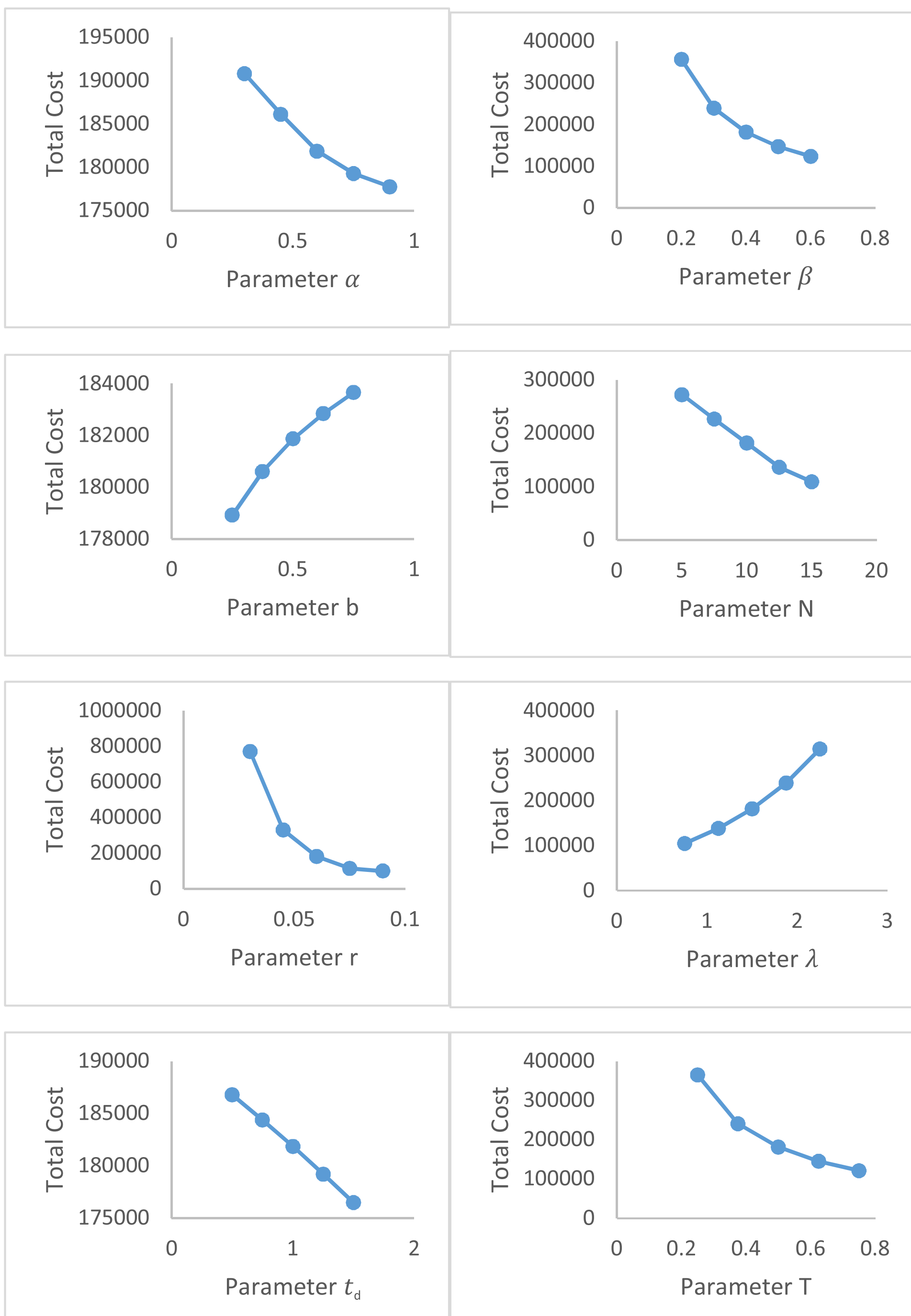


**Fig. 5.** Analysis of the TC vs the parameters.

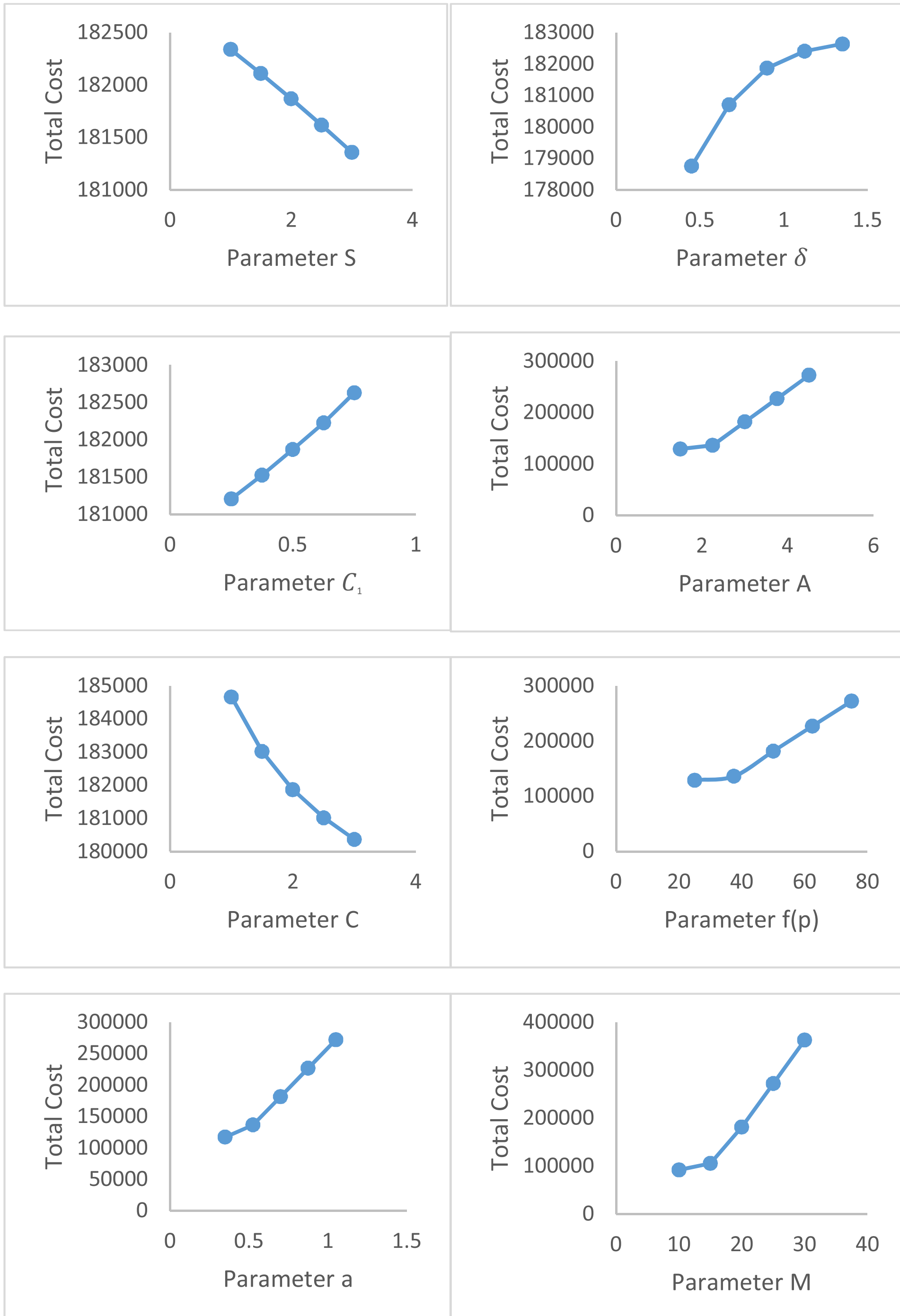


**Fig. 5.** (*continued*).

the importance of strategic marketing decisions. Managers should assess the cost-benefit trade-offs of advertising initiatives to allocate resources effectively. By targeting high-impact advertising channels and continually evaluating campaign performance, managers can maximize returns and drive demand efficiently.

iv. **Cycle time optimization**: Sensitivity analysis emphasizes the significance of cycle time in determining total costs. Managers must streamline production and delivery workflows by identifying and addressing supply chain bottlenecks. Implementing lean manufacturing principles, investing in automation, and improving coordination can reduce cycle times, leading to quicker inventory turnover and lower holding costs.

v. **Shortage cost management**: Understanding the trade-offs between shortage costs and purchasing costs is crucial for effective inventory management. Managers should implement strategies to minimize shortages while balancing the associated costs. This approach helps optimize overall inventory expenses and ensures a more efficient supply chain.

vi. **Parameter management**: Sensitivity analysis indicates that parameters such as non-instantaneous deterioration time and lost sale parameters significantly impact overall costs. Managers need to carefully manage and adjust these parameters to achieve optimal cost-effectiveness. Properly tuning these parameters can lead to better inventory control and reduced expenses.

vii. **Strategic pricing decisions**: The sensitivity of overall costs to pricing parameters highlights the importance of strategic pricing decisions. Managers should carefully consider pricing strategies and adjust pricing parameters to maximize revenue and minimize costs. Effective pricing decisions can enhance profitability and market competitiveness.

## 8. Conclusion and scope

This work proposes a unique inventory model that addresses non-instantaneous deterioration while accounting for the impact of advertisement and selling price-dependent demand in the setting of inflation. Real-world demand dynamics are often shaped by various factors, with selling price and marketing playing pivotal roles. The influence of inflation adds another layer of complexity. The paper undertakes a comparative analysis of fully and partially backlogged shortages in two storage facilities, amalgamating these real-world scenarios into a unified mathematical framework. The results reveal that among the seven models examined, the proposed model emerges with the lowest overall cost. Essentially, this model proves most effective for managing non-instantaneous deteriorating goods with partially backlogged shortages, providing practical insights for inventory optimization in such scenarios. In real life, managing oil inventory is crucial for industries where factors like gradual deterioration, demand fluctuations due to advertising and pricing, and sensitivity to inflation require sophisticated inventory models to minimize costs.

## Real life applications

The proposed strategy outlined above is well-suited for businesses or industries requiring multiple warehouses to manage their inventory effectively. This concept proves particularly beneficial for refined oil products, which experience gradual deterioration due to factors such as oxidation and contamination. Additionally, the sensitivity of oil prices to inflation and other economic variables necessitates careful inventory management. In practical terms, management can enhance advertising efforts to accelerate sales and increase revenue. This real-life application underscores the relevance of the suggested plan in addressing the specific challenges and opportunities associated with managing refined oil inventory across multiple warehouse facilities while considering the impact of inflation.

## Future scope

Subsequent research endeavours in this domain could explore additional dimensions such as stock-dependent demand, trade credit policies, micro-periodic strategies, green supply chain practices, credit period policies, dispatching methodologies & demand. There is abundant room for escalating the existing model to encompass scenarios with multiple buyers interacting with a single supplier, demand dynamics contingent on inventory levels, and other relevant factors. This extended scope could provide a more comprehensive understanding of the intricate dynamics within supply chain management and offer valuable insights for optimizing various policies and strategies in a broader range of contexts.

## CRediT authorship contribution statement

**K Rangarajan:** Writing – review & editing, Supervision, Investigation, Conceptualization. **Anthony Limi:** Writing – original draft, Methodology, Investigation, Conceptualization. **Ehab Ghith:** Writing – review & editing, Methodology, Investigation. **Chiranjibe Jana:** Writing – review & editing, Supervision, Methodology, Investigation, Conceptualization. **Gerhard-Wilhelm Weber:** Writing – review & editing, Supervision, Investigation. **Tarik Lamoudan:** Writing – review & editing, Methodology, Investigation. **Abdelaziz A. Abdelhamid:** Writing – review & editing, Supervision, Methodology.

## Declaration of Competing Interest

The authors have declare that they no conflict of interest for publication of this paper.

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